RESEARCH PAPERS

Local times of N-parameter Gaussian processes*

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Abstract An N-parameter Gaussian stationary process $X=\{X(t): t\in R_+^N\}$ is introduced and the existence and joint continuity of its local times is presented. And the moments of local times are estimated. Furthermore moduli of continuity and large increment results for the local times are established.

Keywords: N-**parameter Gaussian process, local time, moduli of continuity, large increments.**

1 Introduction and statement of the main results

The parameter space is $R_+^N = [0, \infty)^N$, throughout. A typical parameter $t\in R_+^N$ is written as $t=(t_1, \cdots, t_N)$ coordinatewise. There is a natural partial order "$\preceq$" on R_+^N. Namely, $s\preceq t$ if and only if $s_l\leqslant t_l$ for all $l=1, \cdots, N$. When $s\preceq t$, we define the closed interval

$$[s, t] = \prod_{l=1}^{N}[s_l, t_l].$$

Throughout, we will let $\mathscr{A}$ denote the class of all N-dimensional closed intervals $I\subset(0, \infty)^N$ with the form $I=[s, t]$, where $s\preceq t$ and both are in $(0, \infty)^N$. We always write λ_m for Lebesgue's measure on R^m, and use $\langle\circ, \circ\rangle$ and $|\circ|$ to denote the ordinary scalar product and the Euclidean norm respectively.

Definition 1. Let $X=\{X(t); t\in R_+^N\}$ be a real-valued N-parameter stochastic process. For any Borel set A in the line R and $T\subset R_+^N$, let the occupation time

$$H(A, T) = \lambda_N\{s; s\in T, X(s)\in A\}. \quad (1)$$

If, for any fixed T, $H(\circ, T)$ is absolutely continuous with respect to Lebesgue's measure λ_N in R_+^N then its Radon-Nikodym derivative is called the local time of X on T, denoted by $L(x, T)$. It follows from the definition that

$$L(x, S)\leqslant L(x, T) \text{ for } S\subseteq T\subseteq R_+^N, \quad (2)$$

$$H(A, T) = \int_A L(x, T)\mathrm{d}x \quad (3)$$

and

$$H(A, [0, t+h]) - H(A, [0, t]) = \int_A (L(x, [0, t+h]) - L(x, [0, t]))\mathrm{d}x,$$
$$t, t+h\in R_+^N.$$

Definition 2. A real-valued N-parameter stochastic process $X=\{X(t); t\in R_+^N\}$ is called Gaussian process with stationary increments, if for $t\in R_+^N$, $X(t)$ is a Gaussian random variable, and for any $[t, t+h]\subseteq R_+^N$

$$X([t, t+h]) \overset{d}{=} X(h).$$

The local times of single parameter Gaussian process with stationary increments have been studied by Csörgő et al.[1]. In this paper, we study local times of N-parameter Gaussian process with stationary increments under the following assumption:

$$\mathrm{Cov}(X(s), X(t)) = \prod_{j=1}^{N}\frac{1}{2}(\sigma_j^2(|s_j|) + \sigma_j^2(|t_j|) - \sigma_j^2(|s_j - t_j|)), \quad (4)$$

where $\sigma_j^2(l)$, $j=1, 2, \cdots$, are non-decreasing and concave on $(0, \infty)$. Some common and important N-parameter Gaussian processes satisfy (4), such as N-

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parameter Brownian motion, the Kiefer process and N-parameter fractional Brownian motion. Condition (4) implies that

$$\sigma^2(t) := EX^2(t) = \prod_{j=1}^{N} \sigma_j^2(t_j). \tag{5}$$

In the case of a single parameter standard Wiener process, there has been a great amount of elegant work on its local time. Hawkes[2] showed the moduli of continuity of $L(x,[0,t])$ on t

$$\lim_{h\to 0} \sup_{0\leqslant t\leqslant 1-h} \frac{L(0,[0,t+h])-L(0,[0,t])}{\{h\log h^{-1}\}^{1/2}} = 1 \quad \text{a.s.}, \tag{6}$$

while Perkins[3] obtained

$$\limsup_{h\to 0} \sup_{0\leqslant t\leqslant 1-h} \sup_{-\infty<x<\infty} \frac{L(x,[0,t+h])-L(x,[0,t])}{\{2h\log h^{-1}\}^{1/2}} = 1 \quad \text{a.s.} \tag{7}$$

Csáki et al.[4] proved that one can replace $\limsup_{h\to 0}$ by $\lim_{h\to 0}$ in (7) and they obtained also the following results: let $0< a(t)\leqslant t$ be a non-decreasing function of $t\geqslant 0$, and assume that $a(t)/t$ is non-increasing, then

$$\limsup_{t\to\infty} \sup_{0\leqslant s\leqslant t-a(t)} \frac{L(x,[0,s+a(t)])-L(x,[0,s])}{\{a(t)(\log t/a(t)+2\log\log t)\}^{1/2}} = 1 \quad \text{a.s.} \tag{8}$$

for each $x\in R$, and

$$\limsup_{t\to\infty} \sup_{0\leqslant s\leqslant t-a(t)} \sup_{-\infty<x<\infty} \frac{L(x,[0,s+a(t)])-L(x,[0,s])}{\{a(t)(\log t/a(t)+2\log\log t)\}^{1/2}} = 1 \quad \text{a.s.} \tag{9}$$

Moreover, if we also assume that $\lim_{t\to\infty}\left(\log \dfrac{t}{a(t)}\right) \Big\backslash \log\log t = \infty$, then $\limsup_{t\to\infty}$ can be replaced by $\lim_{t\to\infty}$ in Eqs. (8) and (9). Taking $a(t)=t$ in both (8) and (9), we obtain the law of the iterated logarithm, proved by Kesten[5] for local time $L(\cdot,\cdot)$ of a Wiener process:

$$\limsup_{t\to\infty} \frac{L(x,[0,t])}{(2t\log\log t)^{1/2}} = \limsup_{t\to\infty} \sup_{-\infty<x<\infty} \frac{L(x,[0,t])}{(2t\log\log t)^{1/2}} = 1 \quad \text{a.s.} \tag{10}$$

For local time $L(x,T)$ of a single parameter Gaussian process $X=\{X(t);t\geqslant 0\}$, Csörgő et al.[1] showed some results, which are analogous to the above (6), (7) and (8).

The main objective of this paper is to study the existence and joint continuity of the local times of an N-parameter Gaussian process. The results we obtained are analogous to those of Csörgő et al.[1].

First we point out existence and joint continuity of the local times of an N-parameter Gaussian process. Berman[6] and Gemen[7] showed the existence of the joint continuity of local times of a Gaussian process in the one-parameter case, and by a similar procedure we have the following theorem.

Theorem 1. Let $X=\{X(t);\, t\in R_+^N\}$ be an N-parameter Gaussian process with mean zero and stationary increments. If

$$\int_0^{t_1}\cdots\int_0^{t_N} \frac{\mathrm{d}s_1\cdots\mathrm{d}s_N}{\sigma_1(s_1)\cdots\sigma_j(s_N)} < N \text{ for any } t\in R_+^N,$$

then the local time of X exists and if $\sigma_j^2(h_j)$ is continuous and concave for $0\leqslant h_j\leqslant 1$, $j=1,\cdots,N$, then $L(x,T)$ is jointly continuous almost surely.

Theorem 2. Let $X=\{X(t);\, t\in R_+^N\}$ be a Gaussian process with mean zero and stationary increments. Assume that condition (4) is satisfied.

(i) Assume that $X(0)=0$ and $\sigma_j^2(l)$ is non-decreasing and concave on $(0,h_{0j})$ satisfying

$$\sigma_j(ah_j)\geqslant c_0^j a^{\alpha_j}\sigma_j(h_j), \quad 0\leqslant h_j\leqslant h_{0j} \tag{11}$$

for some $0<a<1$, $0<\alpha_j\leqslant 1/2$, c_0^j, $j=1,2,\cdots,N$. Put $c_0=\prod_{j=1}^{N} c_0^j$, $\alpha=\sum_{j=1}^{N}\alpha_j$. Then

$$E(L(x,[t,t+h]))^m \leqslant \left(\frac{1}{2\pi}\right)^{m/2} 2^{mN} \exp\left(-\frac{x^2}{2\sigma^2(t+h)}\right) \cdot \frac{16^{Nm}(m!)^{\alpha}}{c_0^m\sigma^m(h)} \lambda([0,h])^m \tag{12}$$

for each integer $m\geqslant 1$, $0<h\leq h_0=(h_{01},\cdots,h_{0N})\in R_+^N$ and $x\in R$.

If, additionally, X is stationary with $EX^2(0)=1$ instead of $X(0)=0$ and $\sigma_j^2(h_{0j})\leqslant 2$, then

$$E(L(x,[t,t+h]))^m \leqslant \left(\frac{1}{2\pi}\right)^{m/2} 2^{mN} \exp\left(-\frac{x^2}{2^{N+1}}\right) \cdot \frac{16^{Nm}(m!)^{\alpha}}{c_0^m\sigma^{m-1}(h)} \lambda([0,h])^m \tag{13}$$

for each integer $m \geqslant 2$, $0 < h \leq h_0$, $x \in R$.

(ii) Assume that $X(0) = 0$ and $\sigma_j^2(l)$ is non-decreasing and concave on $(0, 1)$, satisfying

$$\sigma_j(ah_j) \geqslant c_0^j a^{\alpha_j} \sigma_j(h_j), \quad 0 \leqslant h_j \leqslant 1 \quad (14)$$

for some $0 < a < 1$, $0 < \alpha_j \leqslant 1/2$, $c_0^j > 0$, $j = 1, \cdots, N$. Then

$$E \sup_x (L(x, [t, t+h]))^m \leqslant 32 \times 12^m \frac{172^{Nm} \sigma^{-4/3}(1)(m!)^{N+\alpha}}{c_0^{m+8/3} \sigma^m(h)} \cdot \lambda([0, h])^{m-4\alpha/3} \prod_{j=1}^N (1 + \sigma_j(2)) \quad (15)$$

for each even integer $m \geqslant 4$, $0 < h \leq 1$, $0 < t \leq 1$.

If, additionally, X is stationary with $EX^2(0) = 1$ instead of $X(0) = 0$, then

$$E \sup_x (L(x, [t, t+h]))^m \leqslant 64 \times 12^m \frac{86^{Nm} \sigma^{-1/3}(1)(m!)^{N+\alpha}}{c_0^{m+8/3} \sigma^m(h)} \cdot \lambda([0, h])^{m-\alpha/3}. \quad (16)$$

Next we give results on large increments and moduli of continuity of the local times of an N-parameter Gaussian process.

Theorem 3. Let $a(t) = (a^1(t), \cdots, a^N(t))$ and $b(t) = (b^1(t), \cdots, b^N(t))$ be non-negative functions of $t \in R_+^N$. For $j = 1, \cdots, N$, put $a_j^* = \sup_{t \in R_+^N} a^j(t)$. Let $X = \{X(t); t \in R_+^N\}$ be a Gaussian process with $X(0) = 0$, mean zero and stationary increments. Assume that condition (4) is satisfied with $\sigma_j(l)$ satisfying the conditions in (i) of Theorem 2 with $h_{0j} = a_j^*$, $j = 1, \cdots, N$, and

$$\frac{1 + \lambda_N([0, b(t)])}{\lambda_N([0, a(t)])} \to \infty \quad \text{as} \quad \lambda_N([0, t]) \to \infty. \quad (17)$$

Then

$$\limsup_{\lambda_N([0,t]) \to \infty} \sup_{0 \leq s \leq b(t)} \frac{L(x, [s, s+a(t)])}{\lambda_N([0, a(t)]) \beta_t^{\alpha} / \sigma(a(t))} \leqslant \frac{16^N}{c_0} \left(\frac{2}{\alpha} \right)^{\alpha} \quad \text{a.s.} \quad (18)$$

for any $x \in R$ where

$$\beta_t = \log \frac{\lambda_N([0, b(t)])}{\lambda_N([0, a(t)])} + N \log\log \left(\lambda_N([0, a(t)]) + \frac{1}{\lambda_N([0, a(t)])} \right). \quad (19)$$

Corollary 1. Let $X = \{X(t); t \in R_+^N\}$ be a Gaussian process satisfying the conditions in Theorem 1 with $h_{0j} = 1$ instead of $h_{0j} = a_j^*$, $j = 1, \cdots, N$. Then

$$\limsup_{\lambda_N([0,h]) \to 0} \frac{L(0, [0, h])}{\lambda_N([0, h])(N \log\log(1/\lambda_N([0, h])))^{\alpha} / \sigma(h)} \leqslant \frac{16^N}{c_0} \left(\frac{2}{\alpha} \right)^{\alpha} \quad \text{a.s.}$$

$$\limsup_{\lambda_N([0,h]) \to 0} \sup_{0 \leq s \leq 1} \frac{L(0, [s, s+h])}{\lambda_N([0, h])(N \log(1/\lambda_N([0, h])))^{\alpha} / \sigma(h)} \leqslant \frac{16^N}{c_0} \left(\frac{2}{\alpha} \right)^{\alpha} \quad \text{a.s.}$$

Corollary 2. Let $Z = \{Z(t); t \in R_+^N\}$ be an N-parameter fractional Wiener process of order $\tau = (\alpha_1, \cdots, \alpha_N)$, $0 < \alpha_j \leqslant 1/2$, i.e. a centered Gaussian process with stationary increments and $EX([0, h])^2 = \prod_{j=1}^N h_j^{2\alpha_j}$. Then

$$\limsup_{\lambda_N([0,h]) \to 0} \frac{L(0, [0, h])}{(\lambda_N([0, h]))^{1-\alpha} (N \log\log(1/\lambda_N([0, h])))^{\alpha}} \leqslant \frac{16^N}{c_0} \left(\frac{2}{\alpha} \right)^{\alpha} \quad \text{a.s.}$$

$$\limsup_{\lambda_N([0,h]) \to 0} \sup_{0 \leq s \leq 1} \frac{L(0, [s, s+h])}{(\lambda_N([0, h]))^{1-\alpha} (N \log(1/\lambda_N([0, h])))^{\alpha}} \leqslant \frac{16^N}{c_0} \left(\frac{2}{\alpha} \right)^{\alpha} \quad \text{a.s.}$$

Theorem 4. Let $b(h) = (b^1(h), \cdots, b^N(h))$ be a function of $h \in R_+^N$ and $X = \{X(t); t \in R_+^N\}$ be a stationary N-parameter Gaussian process with mean zero, stationary increments and $EX^2(0) = 1$. Assume that condition (4) is satisfied with $\sigma_j^2(l)$, $j = 1, \cdots, N$ satisfying the conditions in (ii) of Theorem 2. Then

$$\limsup_{\lambda_N([0,h]) \to 0} \sup_{0 \leq s \leq b(h)} \frac{L(x, [s, s+h])}{\lambda_N([0, h]) \gamma_h^{\alpha} / \sigma(h)} \leqslant \frac{16^N}{c_0} \left(\frac{2}{\alpha} \right)^{\alpha} \quad \text{a.s.} \quad (20)$$

for any $x \in R$, where

$$\gamma_h = \log \left(1 + \frac{\lambda_N([0, h]) + \lambda_N([0, b(h)])}{\lambda_N([0, h])} \cdot \sigma(h) \log^{3/2} \frac{1}{\lambda_N([0, h])} \right). \quad (21)$$

Corollary 3. Under the assumptions of Theorem 4, we have

$$\limsup_{\lambda_N([0,h]) \to 0} \sup_{0 \leqslant \lambda_N([0,s]) \leqslant (\lambda_N([0,h]))^{\theta}} \frac{L(x, [s, s+h])}{\lambda_N([0, h]) \beta_h / \sigma(h)}$$

$$\leqslant \frac{16^N}{c_0}\left(\frac{2}{\alpha}\right)^{\alpha} \quad \text{a.s.}$$

and

$$\limsup_{\lambda_N([0,h])\to 0} \frac{L(x,[0,h])}{\lambda_N([0,h])\sigma(h)^{\alpha-1}\log^{2\alpha}(1/\lambda_N([0,h]))} = 0 \quad \text{a.s.}$$

for each $x \in R$, $0 \leqslant \theta \leqslant 1$, where

$$\beta_h = \log^{\alpha}(1+(\lambda_N([0,h]))^{\theta-1} \circ \sigma(h)\log^{3/2}(1/\lambda_N([0,h]))).$$

$$\limsup_{\lambda_N([0,h])\to 0} \sup_{0\leqslant \lambda_N([0,s])\leqslant(\lambda_N([0,h]))^{\theta}} \frac{L(x,[s,s+h])}{(\lambda_N([0,h]))^{1+\alpha(\theta-2+\alpha)}\log^{2\alpha}(1/\lambda_N([0,h]))} = 0 \quad \text{a.s.}$$

for each $x \in R$ and $0 \leqslant \theta \leqslant 1$.

Corollary 4. Let $X = \{X(t);\ t \in R_+^N\}$ be a stationary Gaussian process with mean zero, stationary increments and $EX^2(0)=1$. Assume that $\sigma_j^2(l)$ is non-decreasing, continuous and concave on (0, 1), satisfying $c_1^j h_j^{\alpha_j} \leqslant \sigma_j(h_j) \leqslant c_0^j h_j^{\alpha_j}$ for some $0<\alpha_j\leqslant 1/2$, $0<c_1^j\leqslant c_0^j$ and all $0<h_j\leqslant 1$, $j=1,2,\cdots,N$. Then

Theorem 5. Let $X=\{X(t);\ t\in R_+^N\}$ be an N-parameter Gaussian process with mean zero, stationary increments and $X(0)=0$. Assume that $\sigma_j^2(l)$, $j=1,\cdots,N$, satisfy the conditions in Theorem 4. Then

$$\limsup_{\lambda_N([0,h])\to 0} \sup_{0\leqslant s\leqslant 1} \sup_{-\infty<x<\infty} \frac{L(x,[s,s+h])}{\lambda_N([0,h])(\log(1/\lambda_N([0,h])))^{\alpha+N}/\sigma(h)} \leqslant \frac{12\times 172^N}{c_0}\left(2+\frac{8}{3}\alpha\right)^{\alpha+N} \quad \text{a.s.} \tag{22}$$

Theorem 6. Let $X=\{X(t);\ t\in R_+^N\}$ be an N-parameter stationary Gaussian process with mean zero, stationary increments and $EX^2(0)=1$. Assume that $\sigma_j^2(l)$, $j=1,\cdots,N$, satisfy the conditions in Theorem 4. Then (22) holds.

2 Proofs of theorems

In order to prove Theorem 2, we need some lemmas. The following lemma is a version of Lemma 3.5 in Ref. [1] for the N-parameter case.

Lemma 1. Let $X=\{X(t);\ t\in R_+^N\}$ be a Gaussian process with mean zero and stationary increments. Let $m\geqslant 1$ be an integer and $\boldsymbol{R}(s_1,\cdots,s_m)$ be the covariance matrix of $X(s_1),\cdots,X(s_m)$. Then for any $x\in R$, t, $h\in R_+^N$

$$E(L(x,[t,t+h]))^m = \left(\frac{1}{2\pi}\right)^{m/2}\int_{[t,t+h]}\cdots\int_{[t,t+h]}\exp\left\{-\frac{x^2}{EX^2(s_1)}\right\} \circ |\boldsymbol{R}(s_1,\cdots,s_n)|^{-1/2}\mathrm{d}s_1\cdots\mathrm{d}s_m, \tag{23}$$

where $|\boldsymbol{R}|$ denotes the determinant of $\boldsymbol{R}$.

Lemma 2. Let $X=\{X(t);\ t\in R_+^N\}$ be an N-parameter Gaussian process with stationary increments and covariance function (4) and let $X^j=\{X^j(t),\ t\in R_+\}$, $j=1,\cdots,N$, be a single parameter Gaussian process with mean zero, stationary increments and $\mathrm{Var}X^j(l)=\sigma_j^2(l)$, $j=1,\cdots,N$, for $l\in R_+$. Then for any integer $n\geqslant 2$ and $s_k=(s_{k1},\cdots,s_{kN})$, $k=1,\cdots,n$, we have

$$|\boldsymbol{R}(s_1,\cdots,s_n)| \geqslant \prod_{j=1}^{N}|\boldsymbol{R}_j(s_{1j},\cdots,s_{nj})|, \tag{24}$$

where $\boldsymbol{R}(s_1,\cdots,s_n)$ and $\boldsymbol{R}_j(s_{1j},\cdots,s_{nj})$ are covariance matrices of $(X(s_1),\cdots,X(s_n))$ and $(X^j(s_{1j}),\cdots,X^j(s_{nj}))$ respectively.

Proof. Recall that the Hadamard product of two $n\times n$ matrices $\boldsymbol{A}=(a_{ij})$ and $\boldsymbol{B}=(b_{ij})$ is an $n\times n$ matrix defined as $\boldsymbol{A}\circ\boldsymbol{B}=(a_{ij}b_{ij})$. A classical theorem of Oppenheim (cf. Horn et al.[8]) asserts that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are positive semidefinite Hermitian matrices, then

$$|\boldsymbol{A}\circ\boldsymbol{B}| \geqslant |\boldsymbol{A}|\circ|\boldsymbol{B}|. \tag{25}$$

By condition (4) we see that the covariance matrix $\boldsymbol{R}(s_1,\cdots,s_n)$ is the Hadamard product of the covariance matrices $\boldsymbol{R}_j(s_{1j},\cdots,s_{nj})$, $j=1,\cdots,N$. Hence (24) is true.

Proof of Theorem 2. Recalling (5) and using Lemmas 1 and 2 and Lemma 3.3 in [1] we have

$$E(L(x,[t,t+h]))^m \leqslant \left(\frac{1}{2\pi}\right)^{m/2}\int_{[t,t+h]}\cdots\int_{[t,t+h]}\exp\left(\frac{-x^2}{2EX^2(s_1)}\right) \circ |\boldsymbol{R}(s_1,\cdots,s_m)|^{-\frac{1}{2}}\mathrm{d}s_1\cdots\mathrm{d}s_m$$

$$\leqslant \left(\frac{1}{2\pi}\right)^{m/2}\exp\left(-\frac{x^2}{2\sigma^2(t+h)}\right)$$

$$\times \prod_{j=1}^{N} \int_{[t_j, t_j+h_j]} \cdots \int_{[t_j, t_j+h_j]}$$
$$\circ \left| \boldsymbol{R}_j(s_{1j}) \cdots s_{mj} \right|^{-\frac{1}{2}} \mathrm{d}s_{1j} \cdots \mathrm{d}s_{mj}$$

$$\leqslant \left(\frac{1}{2\pi}\right)^{m/2} 2^{mN} \exp\left(-\frac{x^2}{2\sigma^2(t+h)}\right)$$
$$\times \prod_{j=1}^{N} \int_{[t_j, t_j+h_j]} \cdots \int_{[t_j, t_j+h_j]}$$
$$\circ \frac{1}{\sigma_j(s_{1j})\sigma_j(s_{2j}-s_{2j})\cdots\sigma_j(s_{mj}-s_{m-1,j})}$$
$$\circ \mathrm{d}s_{1j} \cdots \mathrm{d}s_{mj}$$
$$\leqslant \left(\frac{1}{2\pi}\right)^{m/2} 2^{mN} \exp\left(-\frac{x^2}{2\sigma^2(t+h)}\right)$$
$$\times \prod_{j=1}^{N} \frac{m!h_j^m}{c_0^m \sigma(h_j)^m} \int \cdots \int_{0< l_1 \leqslant \cdots \leqslant l_m \leqslant 1}$$
$$\circ \frac{1}{l_1^\alpha (l_2-l_1)^\alpha \cdots (l_m-l_{m-1})^\alpha} \mathrm{d}l_1 \cdots \mathrm{d}l_m$$
$$\leqslant \left(\frac{1}{2\pi}\right)^{m/2} 2^{mN} \exp\left(-\frac{x^2}{2\sigma^2(t+h)}\right)$$
$$\circ \prod_{j=1}^{N} \frac{(16h_j)^m}{(c_0^j)^m \sigma_j(h_j)^m} (m!)^{\alpha_j}$$
$$\leqslant \left(\frac{1}{2\pi}\right)^{m/2} 2^{mN} \exp\left(-\frac{x^2}{2\sigma^2(t+h)}\right)$$
$$\circ \frac{16^{Nm}(m!)^\alpha}{c_0^m \sigma^m(h)} \lambda([0, h])^m, \tag{26}$$

here the fourth inequality follows in the same way as in the proof of (3.10) in [1]. (12) is proved.

The proof of (13) is similar to that of (12) by using Lemma 3.4 instead of Lemma 3.3 in [1].

For an N-parameter Gaussian process $X(t)$, we have similar results in Lemmas 4.1 and 4.2 in [1]. Using Lemma 2 again and along the lines of the proofs of Lemmas 4.3 and 4.4 in [1] and (12), we can prove (15) and (16).

Lemma 3. Let ξ be a non-negative random variable. Assume that

$$E\xi^m \leqslant C(m!)^\alpha \tag{27}$$

for some $C>0$, $\alpha>0$ and each $m \geqslant 2$. Then

$$P\{\xi > y\} \leqslant \frac{K_\alpha C}{(\exp(y^{1/\alpha}/4)-1)^{2\alpha}} \tag{28}$$

for any $y>0$, where K_α is a positive constant depending only on α.

The proof can be found in [1].

Lemma 4. Under the assumptions of Theorem 3, we have

$$P\left\{ L(x, [t, t+h]) \geqslant \frac{2^{N-1/2}}{\pi^{1/2}} \frac{16^N \lambda([0, h])}{c_0 \sigma(h)} y \right\}$$
$$\leqslant \frac{K_\alpha \exp\{-x^2/2\sigma^2(t+h)\}}{(\exp(y^{1/\alpha}/4)-1)^{2\alpha}} \tag{29}$$

for any $0 \prec h \preceq a^*$, $y>0$, $x \in R$.

The proof follows immediately from (12) and Lemma 3.

Proof of Theorem 3.

Let $1<\theta<\frac{5}{4}$. For $j=1, \cdots, N$, define

$$A_{k_j}^j = \{t; \theta^{k_j} < d^j(t) \leqslant \theta^{k_j+1}\},$$
$$-\infty < k_j < \infty,$$
$$A_{k_j, l_j}^j = \{t; \theta^{l_j} \leqslant b^j(t)/d^j(t) < \theta^{l_j+1}, t \in A_{k_j}^j\},$$
$$l_j = 0, 1, \cdots.$$

It is easy to see that

$$\beta_t \geqslant \beta_{k,l} := \log \theta^l + N \log\log \theta^{|k|}$$

for any $t \in \bigcap_{j=1}^{N} A_{k_j, l_j}^j$ with $k = \sum_{j=1}^{N} k_j$, $l = \sum_{j=1}^{N} l_j$, where $\theta>1$ is a constant. Put $\boldsymbol{K}=(k_1, \cdots, k_N)$, $L=(l_1, \cdots, l_N)$ and $a_K=(\theta^{k_1+1}, \cdots, \theta^{k_N+1})$, $b_{K,L}=(\theta^{l_1+k_1+2}, \cdots, \theta^{l_N+k_N+2})$, $\theta^{L+1}=(\theta^{l_1+1}, \cdots, \theta^{l_N+1})$ and define $ab=(a_1b_1, \cdots, a_Nb_N)$, for $a=(a_1, \cdots, a_N)$, $b=(b_1, \cdots, b_N)$. Noting that $L(x, T)$ is non-decreasing in T for each fixed x, we have

$$\limsup_{\lambda_N([0,t])\to\infty} \sup_{0 \preceq s \preceq b(t)} \frac{L(x, [s, s+a(t)])}{\lambda_N([0, a(t)])\beta_t^\alpha / \sigma(a(t))}$$
$$\leqslant \limsup_{|k+r|\to\infty} \sup_{l_j \geqslant r_j, j=1,\cdots,N} \sup_{t \in \bigcap_{j=1}^N A_{k_j,l_j}^j} \sup_{0 \preceq s \preceq b(t)} \frac{\sigma(a(t))(L(x, [s, s+a(t)]))}{\lambda_N([0, a(t)])\beta_t^\alpha}$$
$$\leqslant \limsup_{|k+r|\to\infty} \sup_{l_j \geqslant r_j, j=1,\cdots,N} \sup_{t \in \bigcap_{j=1}^N A_{k_j,l_j}^j} \sup_{0 \preceq s \preceq b_{K,L}} \frac{\sigma(a_K) L(x, [s, s+a_K])}{\theta^k \beta_{k,l}^\alpha}$$
$$\leqslant \limsup_{|k+r|\to\infty} \sup_{l_j \geqslant r_j, j=1,\cdots,N} \max_{0 \leqslant M \leqslant \theta^{L+1}} \frac{\sigma(a_K) L(x, [Ma_K, (M+1)a_K])}{\theta^k \beta_{k,l}^\alpha}, \tag{30}$$

where $r=\sum_{j=1}^{N} r_j$, $M=(m_1, \cdots, m_N)$, $m_j=0, 1, \cdots, j=1, \cdots, N$, and $M+1=(m_1+1, \cdots, m_N+1)$. Applying Lemma 4, we have

$$\begin{aligned} &P_{r_1, \cdots, r_N; k_1, \cdots, k_N} \\ &:= P\left\{\sup_{l_j \geqslant r_j, j=1, \cdots, N} \max_{0 \leqslant M \leqslant \theta^{L+1}} \frac{\sigma(a_K) L(x, [Ma_K, (M+1)a_K])}{\theta^k \beta_{k,l}^{\alpha}} \geqslant \theta^N \left(\frac{2\theta}{\alpha}\right)^{\alpha} \frac{16^N}{c_0}\right\} \\ &\leqslant \sum_{l_1=r_1}^{\infty} \cdots \sum_{l_N=r_N}^{\infty} \sum_{m_1=0}^{[\theta^{l_1}]} \cdots \sum_{m_N=0}^{[\theta^{l_N}]} C_1 e^{-\theta \beta_{k,l}} \\ &\leqslant C_2 \sum_{l_1=r_1}^{\infty} \cdots \sum_{l_N=r_N}^{\infty} \theta^{-l(\theta-1)} \log^{-\theta N} \theta^{|k|} \\ &\leqslant C_3 \theta^{-r(\theta-1)} (|k|+1)^{-\theta N}, \end{aligned} \tag{31}$$

where C_j, $j=1, 2, 3$, are constants depending only on θ and α. Note that

$$\sum_{r_1=0}^{\infty} \cdots \sum_{r_N=0}^{\infty} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} P_{r_1, \cdots, r_N; k_1, \cdots, k_N} < \infty.$$

It follows by the Borel-Cantelli lemma that

$$\limsup_{\lambda_N([0,t]) \to \infty} \sup_{0 \leqslant s \leqslant b(t)} \frac{L(x, [s, s+a(t)])}{\lambda_N([0, a(t)]) \beta_t^{\alpha} / \sigma(a(t))} \leqslant \theta^N \left(\frac{2\theta}{\alpha}\right)^{\alpha} \frac{16^N}{c_0} \quad \text{a.s.}$$

By arbitrariness of θ, we obtain (18). The proof of Theorem 3 is completed.

Similar to Lemma 4, a combination of Lemma 3 with (13) implies the following lemma.

Lemma 5. If (13) holds true, then

$$P\left\{L(x, [t, t+h]) \geqslant \left(\frac{1}{2\pi}\right)^{1/2} \frac{32^N \lambda_N([0, h])}{c_0 \sigma(h)} y\right\} \leqslant \frac{K_\alpha \exp(-x^2/2^{N+1}) \sigma(h)}{(\exp(y^{1/\alpha}/4)-1)^{2\alpha}} \tag{32}$$

for each $0 < h \leqslant h_0$, $t \in R_+^N$, $y > 0$, $x \in R$.

Using Lemma 5 instead of Lemma 4, proceeding along the lines of the proof of Theorem 3, we obtain Theorem 4.

The proofs of Theorem 5 and Theorem 6 are based on the following two lemmas, which are similar to that of Lemma 4 by using (15) (resp. (16)) instead of (12).

Lemma 6. Under the assumptions of Theorem 5, we have

$$P\left\{\sup_x (L(x, [t, t+h])) > 12 \frac{172^N \lambda_N([0, h])}{c_0 \sigma(h)} y\right\} \leqslant K_\alpha C_1 (\lambda_N([0, h]))^{-4\alpha/3} \exp(-y^{1/(N+\alpha)}/2) \tag{33}$$

for each $y > 1$, $0 \leqslant t, h \leqslant 1$, where

$$C_1 = 32\sigma^{-4/3}(1) c_0^{8/3} \prod_{j=1}^{N} (1+\sigma_j(2)).$$

Lemma 7. Under the assumptions of Theorem 6, we have

$$P\left\{\sup_x (L(x, [t, t+h])) > 12 \frac{86^N \lambda_N([0, h])}{c_0 \sigma(h)} y\right\} \leqslant K_\alpha C_2 (\lambda_N([0, h]))^{-\alpha/3} \exp(-y^{1/(N+\alpha)}/2) \tag{34}$$

for each $y > 1$, $0 \leqslant t, h \leqslant 1$, where

$$C_2 = 64 c_0^{-8/3} \sigma^{-1/3}(h).$$

Proof of Theorem 5. For integers k_j, $j=1, \cdots, N$, let $K=(k_1, \cdots, k_N)$, $k=\sum_{j=1}^{N} k_j$, $\theta^{\pm K}=(\theta^{\pm k_1}, \cdots, \theta^{\pm k_N})$; for integers $m_j \geqslant 0$, $j=1, \cdots, N$, let $M=(m_1, \cdots, m_N)$, $M+1=(m_1+1, \cdots, m_N+1)$ and $1 < \theta < 5/4$. Then

$$\begin{aligned} &\limsup_{\lambda_N([0,h]) \to 0} \sup_{0 \leqslant s \leqslant 1} \sup_x \frac{L(x, [s, s+h])}{\lambda_N([0, h])(\log(1/\lambda_N([0, h])))^{\alpha+N}/\sigma(h)} \\ &\leqslant \limsup_{k \to -\infty} \sup_{\theta^{k_j-1} \leqslant h_j \leqslant \theta^{k_j}} \sup_{0 \leqslant s \leqslant 1} \sup_x \frac{L(x, [s, s+h])}{\lambda_N([0, h])(\log(1/\lambda_N([0, h])))^{\alpha+N}/\sigma(h)} \\ &\leqslant \limsup_{k \to -\infty} \sup_{0 \leqslant M \leqslant \theta^{-K}} \sup_x \frac{L(x, [M\theta^K, (M+1)\theta^K])}{\theta^{k-N}(\log(\theta^{-K}))^{\alpha+N}/\sigma(\theta^K)}. \end{aligned} \tag{35}$$

From (33), we have

$$\begin{aligned} &P\left\{\sup_{0 \leqslant M \leqslant \theta^{-K}} \sup_x \frac{L(x, [M\theta^K, (M+1)\theta^K])}{\theta^{k-N}(\log(\theta^{-k}))^{\alpha+N}/\sigma(\theta^K)} \geqslant 12 \frac{172^N}{c_0}\left(2\theta+\frac{8}{3}\alpha\right)\theta^{N+\alpha}\right\} \\ &\leqslant \sum_{m_1=0}^{[\theta^{-k_1}]} \cdots \sum_{m_N=0}^{[\theta^{-k_N}]} K_\alpha C_1 \theta^{4k\alpha/3} \cdot \exp\left(-\left(\theta+\frac{4}{3}\alpha\right)\log \theta^{-k} 2\right) \\ &\leqslant K\theta^{k(\theta-1)}. \end{aligned}$$

By the Borel-Cantelli lemma, we obtain

$$\limsup_{k\to-\infty}\sup_{0\le M\le\theta^{-k}}\sup_{x}\frac{L(x,[M\theta^{-K},(M+1)\theta^{-K}])}{\theta^{k-N}(\log(\theta^{-k}))^{\alpha+N}/\sigma(\theta^{K})}$$
$$\leqslant 12\frac{172^{N}}{c_0}\left(2\theta+\frac{8}{3}\alpha\right)\theta^{N+\alpha}\quad \text{a. s.} \qquad (36)$$

By (35) and (36) and taking θ near to 1, (22) is proved.

The proof of Theorem 6 is similar to that of Theorem 5 and therefore is omitted.

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